

Notes on the description of join-distributive lattices by permutations

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ABSTRACT. Let L be a join-distributive lattice with length n and width $\text{width}(L) \leq k$. There are two ways to describe L by $k-1$ permutations acting on an n -element set: a combinatorial way given by P.H. Edelman and R.E. Jamison in 1985 and a recent lattice theoretical way of the second author. We prove that these two approaches are equivalent. Also, we characterize join-distributive lattices by trajectories.

Introduction. For $x \neq 1$ in a finite lattice L , let x^* denote the join of upper covers of x . A finite lattice L is *join-distributive* if the interval $[x, x^*]$ is distributive for all $x \in L \setminus \{1\}$. For other definitions, see K. Adaricheva [2], K. Adaricheva, V.A. Gorbunov and V.I. Tumanov [3], and N. Caspard and B. Monjardet [6], see G. Czédli [7, Proposition 2.1 and Remark 2.2] for a recent survey, and see (3) before the proof of Corollary 6 later for a particularly useful variant. The study of (the duals of) join-distributive lattices goes back to R.P. Dilworth [11], 1940. There were a lot of discoveries and rediscoveries of these lattices and equivalent combinatorial structures; see [3], [7], B. Monjardet [17], and M. Stern [18] for surveys. Note that join-distributivity implies semimodularity; the origin of this result is the combination of M. Ward [19] (see also R.P. Dilworth [11, page 771], where [19] is cited) and S.P. Avann [5] (see also P.H. Edelman [13, Theorem 1.1(E,H)], when [5] is recalled).

The *join-width* of L , denoted by $\text{width}(L)$, is the largest k such that there is a k -element antichain of join-irreducible elements of L . As usual, S_n stands for the set of permutations acting on the set $\{1, \dots, n\}$. There are two known ways to describe a join-distributive lattice with join-width k and length n by $k-1$ permutations; our goal is to enlighten their connection. This connection exemplifies that Lattice Theory can be applied in Combinatorics and vice versa. We also give a new characterization of join-distributive lattices.

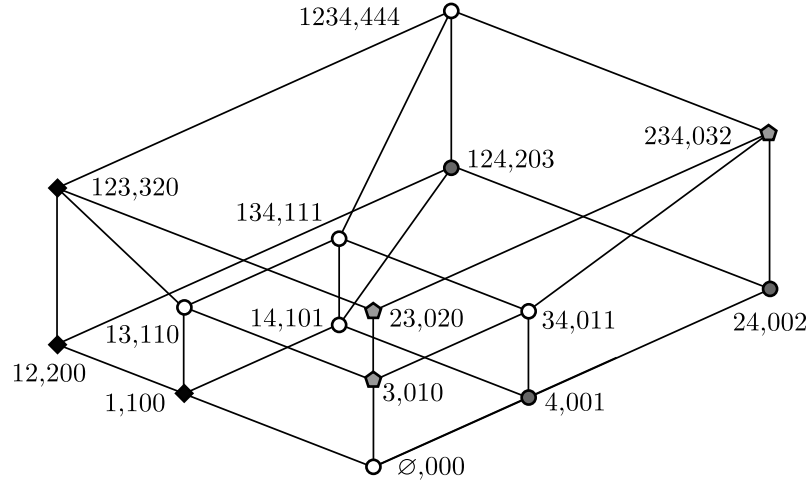
Two constructions. For $n \in \mathbb{N} = \{1, 2, \dots\}$ and $k \in \{2, 3, \dots\}$, let $\vec{\sigma} = \langle \sigma_2, \dots, \sigma_k \rangle \in S_n^{k-1}$. For convenience, $\sigma_1 \in S_n$ will denote the identity permutation. In the powerset join-semilattice $\langle P(\{1, \dots, n\}); \cup \rangle$, consider the subsemilattice $L_{\text{EJ}}(\vec{\sigma})$ generated by

$$\{ \{ \sigma_i(1), \dots, \sigma_i(j) \} : i \in \{1, \dots, k\}, j \in \{0, \dots, n\} \}. \quad (1)$$

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FIGURE 1. An example of $L_{\text{EJ}}(\vec{\sigma})$ and $L_{\text{C}}(\vec{\pi})$

Since it contains \emptyset , $L_{\text{EJ}}(\vec{\sigma})$ is a lattice, the *Edelman-Jamison lattice* determined by $\vec{\sigma}$. Its definition above is a straightforward translation from the combinatorial language of P. H. Edelman and R. E. Jamison [14, Theorem 5.2] to Lattice Theory. Actually, the original version in [14] describes a decomposition of convex geometries.

To present an example, let $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$.

Then $\vec{\sigma} = \langle \sigma_2, \sigma_3 \rangle \in S_4^2$, and $L_{\text{EJ}}(\vec{\sigma})$ is depicted in Figure 1. In the label of an element in $L_{\text{EJ}}(\vec{\sigma})$, only the part before the comma is relevant; to save space, subsets are denoted by listing their elements without commas. For example, 134,111 in the figure stands for the subset $\{1, 3, 4\}$ of $\{1, 2, 3, 4\}$. The chain defined in (1), apart from its top $\{1, 2, 3, 4\}$ and bottom \emptyset , corresponds to the black-filled small squares for $i = 1$, the light grey-filled pentagons for $i = 2$, and the dark grey-filled circles for $i = 3$. Note that $L_{\text{EJ}}(\vec{\sigma})$ consists of all subsets of $\{1, 2, 3, 4\}$ but $\{2\}$.

Next, we recall a related construction from G. Czédli [7]. Given $\vec{\pi} = \langle \pi_{12}, \dots, \pi_{1k} \rangle \in S_n^k$, we let $\pi_{ij} = \pi_{1j} \circ \pi_{1i}^{-1}$ for $i, j \in \{1, \dots, k\}$. Here we compose permutations from right to left, that is, $(\pi_{1j} \circ \pi_{1i}^{-1})(x) = \pi_{1j}(\pi_{1i}^{-1}(x))$. Note that $\pi_{ii} = \text{id}$, $\pi_{ij} = \pi_{ji}^{-1}$, and $\pi_{jt} \circ \pi_{ij} = \pi_{it}$ hold for all $i, j, t \in \{1, \dots, k\}$. By an *eligible $\vec{\pi}$ -tuple* we mean a k -tuple $\vec{x} = \langle x_1, \dots, x_k \rangle \in \{0, 1, \dots, n\}^k$ such that $\pi_{ij}(x_i + 1) \geq x_j + 1$ holds for all $i, j \in \{1, \dots, k\}$ such that $x_i < n$. Note that an eligible $\vec{\pi}$ -tuple belongs to $\{0, 1, \dots, n-1\}^k \cup \{\langle n, \dots, n \rangle\}$ since $x_j = n$ implies $x_i = n$. The set of eligible $\vec{\pi}$ -tuples is denoted by $L_{\text{C}}(\vec{\pi})$. It is a poset with respect to the componentwise order: $\vec{x} \leq \vec{y}$ means that $x_i \leq y_i$ for all $i \in \{1, \dots, k\}$. It is trivial to check that $\langle n, \dots, n \rangle \in L_{\text{C}}(\vec{\pi})$ and that $L_{\text{C}}(\vec{\pi})$ is a meet-subsemilattice of the k -th direct power of the chain $\{0 < 1 < \dots < n\}$.

Therefore, $L_C(\vec{\pi})$ is a lattice, the $\vec{\pi}$ -*coordinatized lattice*. Its construction is motivated by G. Czédli and E. T. Schmidt [8, Theorem 1], see also M. Stern [18], which asserts that there is a surjective cover-preserving join-homomorphism $\varphi: \{0 \prec \dots \prec n\}^k \rightarrow L$, provided L is semimodular. Then, as it is easy to verify, $u \mapsto \bigvee \{x : \varphi(x) = u\}$ is a meet-embedding of L into $\{0 \prec \dots \prec n\}^k$.

To give an example, let $\pi_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$, $\pi_{13} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$, and let $\vec{\pi} = \langle \pi_{12}, \pi_{13} \rangle \in S_4^2$. Then Figure 1 also gives $L_C(\vec{\pi})$; the eligible $\vec{\pi}$ -tuples are given after the commas in the labels. For example, 23,020 in the figure corresponds to $\langle 0, 2, 0 \rangle$. Note that if $\mu_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$ and $\mu_{13} = \pi_{13}$, then $L_C(\vec{\pi}) \cong L_C(\vec{\mu})$. Furthermore, the problem of characterizing those pairs of members of S_n^k that determine the same lattice is not solved yet if $k \geq 3$. For $k = 2$ the solution is given in G. Czédli and E. T. Schmidt [10]; besides $L_C(\vec{\pi}) \cong L_C(\vec{\mu})$ above, see also G. Czédli [7, Example 5.3] to see the difficulty.

The connection of join-distributivity to $L_{\text{EJ}}(\vec{\pi})$ and $L_C(\vec{\pi})$ will be given soon.

The two constructions are equivalent. For $\langle \gamma_2, \dots, \gamma_k \rangle \in S_n^{k-1}$, we let $\langle \gamma_2, \dots, \gamma_k \rangle^{-1} = \langle \gamma_2^{-1}, \dots, \gamma_k^{-1} \rangle$.

Proposition 1. *For every $\vec{\sigma} \in S_n^{k-1}$, $L_{\text{EJ}}(\vec{\sigma})$ is isomorphic to $L_C(\vec{\sigma}^{-1})$.*

In some vague sense, Figure 1 reveals why $L_{\text{EJ}}(\vec{\sigma})$ could be of the form $L_C(\vec{\pi})$ for some $\vec{\pi}$. Namely, for $x \in L_{\text{EJ}}(\vec{\sigma})$ and $i \in \{1, \dots, k\}$, we can define the i -th coordinate of x as the length of the intersection of the ideal $\{y \in L_{\text{EJ}}(\vec{\sigma}) : y \leq x\}$ and the chain given in (1). However, the proof is more complex than this initial idea.

Proof. Denote $\vec{\sigma}^{-1}$ by $\vec{\pi} = \langle \pi_{12}, \dots, \pi_{1k} \rangle$. Note that $\pi_{11} = \sigma_1^{-1} = \text{id} \in S_n$. For $U \in L_{\text{EJ}}(\vec{\sigma})$ and $i \in \{1, \dots, k\}$, let $U(i) = \max\{j : \{\sigma_i(1), \dots, \sigma_i(j)\} \subseteq U\}$, where $\max \emptyset$ is defined to be 0. We assert that the map

$$\varphi: L_{\text{EJ}}(\vec{\sigma}) \rightarrow L_C(\vec{\pi}), \text{ defined by } U \mapsto \langle U(1), \dots, U(k) \rangle,$$

is a lattice isomorphism. To prove that $\varphi(U)$ is an eligible $\vec{\pi}$ -tuple, assume that $i, j \in \{1, \dots, k\}$ such that $U(i) < n$. Then $\sigma_i(U(i) + 1) \notin U$ yields $\sigma_i(U(i) + 1) \notin \{\sigma_j(1), \dots, \sigma_j(U(j))\}$. However, $\sigma_i(U(i) + 1) \in \{1, \dots, n\} = \{\sigma_j(1), \dots, \sigma_j(n)\}$, and we conclude that $\sigma_i(U(i) + 1) = \sigma_j(t)$ holds for some $t \in \{U(j) + 1, \dots, n\}$. Hence

$$\begin{aligned} \pi_{ij}(U(i) + 1) &= (\pi_{1j} \circ \pi_{i1})(U(i) + 1) = \pi_{1j}(\pi_{i1}(U(i) + 1)) \\ &= \pi_{1j}(\pi_{1i}^{-1}(U(i) + 1)) = \sigma_j^{-1}(\sigma_i(U(i) + 1)) \\ &= \sigma_j^{-1}(\sigma_j(t)) = t \geq U(j) + 1. \end{aligned}$$

This proves that $\varphi(U)$ is an eligible $\vec{\pi}$ -tuple, and φ is a map from $L_{\text{EJ}}(\vec{\sigma})$ to $L_C(\vec{\pi})$. Since $L_{\text{EJ}}(\vec{\sigma})$ is generated by the set given in (1), we conclude

$$U = \bigcup_{i=1}^k \{\sigma_i(1), \dots, \sigma_i(U(i))\}.$$

This implies that U is determined by $\langle U(1), \dots, U(k) \rangle = \varphi(U)$, that is, φ is injective. To prove that φ is surjective, let $\vec{x} = \langle x_1, \dots, x_k \rangle$ be a $\vec{\pi}$ -eligible tuple, that is, $\vec{x} \in L_C(\vec{\pi})$. Define

$$V = \bigcup_{i=1}^k \{\sigma_i(1), \dots, \sigma_i(x_i)\}. \quad (2)$$

(Note that if $x_i = 0$, then $\{\sigma_i(1), \dots, \sigma_i(x_i)\}$ denotes the empty set.) For the sake of contradiction, suppose $\varphi(V) \neq \vec{x}$. Then, by the definition of φ , there exists an $i \in \{1, \dots, k\}$ such that $\sigma_i(x_i + 1) \in V$. Hence, there is a $j \in \{1, \dots, k\}$ such that $\sigma_i(x_i + 1) \in \{\sigma_j(1), \dots, \sigma_j(x_j)\}$. That is, $\sigma_i(x_i + 1) = \sigma_j(t)$ for some $t \in \{1, \dots, x_j\}$. Therefore,

$$\begin{aligned} \pi_{ij}(x_i + 1) &= \pi_{1j}(\pi_{i1}(x_i + 1)) = \pi_{1j}(\pi_{1i}^{-1}(x_i + 1)) = \sigma_j^{-1}(\sigma_i(x_i + 1)) \\ &= \sigma_j^{-1}(\sigma_j(t)) = t \leq x_j, \end{aligned}$$

which contradicts the $\vec{\pi}$ -eligibility of \vec{x} . Thus $\varphi(V) = \vec{x}$ and φ is surjective.

We have shown that φ is bijective. For $\vec{x} \in L_C(\vec{\pi})$, $\varphi^{-1}(\vec{x})$ is the set V given in (2). Thus φ and φ^{-1} are monotone, and φ is a lattice isomorphism. \square

Two descriptions. The following theorem is a straightforward consequence of Theorems 5.1 and 5.2 in P. H. Edelman and R. E. Jamison [14], which were formulated and proved within Combinatorics.

Theorem 2. *Up to isomorphism, join-distributive lattices of length n and join-width at most k are characterized as lattices $L_{\text{EJ}}(\vec{\sigma})$ with $\vec{\sigma} \in S_n^{k-1}$.*

The next theorem was motivated and proved by the second author [7] in a purely lattice theoretical way.

Theorem 3. *Up to isomorphism, join-distributive lattices of length n and join-width at most k are characterized as the $\vec{\pi}$ -coordinatized lattices $L_C(\vec{\pi})$ with $\vec{\pi} \in S_n^{k-1}$.*

Remark 4. Since there is no restriction on $(n, k) \in \mathbb{N} \times \{2, 3, \dots\}$ in Theorems 2 and 3, one might have the feeling that, for a given n , the join-width of a join-distributive lattice of length n can be arbitrarily large. This is not so since, up to isomorphism, there are only finitely many join-distributive lattices of length n .

The statement of Remark 4 follows from the fact that each join-distributive lattice of length n is dually isomorphic to the lattice of closed sets of a convex geometry on the set $\{1, \dots, n\}$, see P. H. Edelman [12, Theorem 3.3] together

with the sixteenth line in the proof of Theorem 1.9 in K. Adaricheva, V.A. Gorbunov and V.I. Tumanov [3]; see also [7, Lemma 7.4], where this is surveyed. The statement also follows, in a different way, from [7, Corollary 4.4].

Remark 5. Obviously, Proposition 1 and Theorem 2 imply Theorem 3 and, similarly, Proposition 1 and Theorem 3 imply Theorem 2. Thus we obtain a new, combinatorial proof of Theorem 3 and a new, lattice theoretical proof of Theorem 2.

Comparison. We can compare Theorems 2 and 3, and the corresponding original approaches, as follows.

In case of Theorem 2, the construction of the lattice $L_{\text{EJ}}(\vec{\sigma})$ is very simple, and a join-generating subset is also given.

In case of Theorem 3, the elements of the lattice $L_C(\vec{\pi})$ are exactly given by their coordinates, the eligible $\vec{\pi}$ -tuples. Moreover, the meet operation is easy, and we have a satisfactory description of the optimal meet-generating subset since it was proved in [7, Lemma 6.5] that

$$\text{Mi}(L_C(\vec{\pi})) = \{ \langle \pi_{11}(i) - 1, \dots, \pi_{1k}(i) - 1 \rangle : i \in \{1, \dots, n\} \}.$$

Characterization by trajectories. For a lattice L of finite length, the set $\{[a, b] : a \prec b, a, b \in L\}$ of prime intervals of L will be denoted by $\text{PrInt}(L)$. For $[a, b], [c, d] \in \text{PrInt}(L)$, we say that $[a, b]$ and $[c, d]$ are *consecutive* if $\{a, b, c, d\}$ is a covering square, that is, a 4-element cover-preserving boolean sublattice of L . The transitive reflexive closure of the consecutiveness relation on $\text{PrInt}(L)$ is an equivalence, and the blocks of this equivalence relation are called the *trajectories* of L ; this concept was introduced for some particular semimodular lattices in G. Czédli and E.T. Schmidt [9]. For distinct $[a, b], [c, d] \in \text{PrInt}(L)$, these two prime intervals are *comparable* if either $b \leq c$, or $d \leq a$. Before formulating the last statement of the paper, it is reasonable to mention that, for any finite lattice L ,

$$L \text{ is join-distributive iff it is semimodular and meet-semidistributive.} \quad (3)$$

This follows from K. Adaricheva, V.A. Gorbunov and V.I. Tumanov [3, Theorems 1.7 and 1.9]; see also D. Armstrong [4, Theorem 2.7] for the present formulation.

Corollary 6. *For a semimodular lattice L , the following three conditions are equivalent.*

- (i) *L is join-distributive.*
- (ii) *L is of finite length, and for every trajectory T of L and every maximal chain C of L , $|\text{PrInt}(C) \cap T| = 1$.*
- (iii) *L is of finite length, and no two distinct comparable prime intervals of L belong to the same trajectory.*

As an interesting consequence, note that each of (ii) and (iii) above, together with semimodularity, implies that L is finite.

Proof of Corollary 6. Since any two comparable prime intervals belong to the set of prime intervals of an appropriate maximal chain C , (ii) implies (iii). So we have to prove that (i) \Rightarrow (iii) and that (iii) \Rightarrow (i); we give two alternative arguments for each of these two implications. Let $n = \text{length } L$.

Assume (i). Then L is semimodular by (3), and it contains no cover-preserving diamond by the definition of join-distributivity. Thus G. Czédli [7, Lemma 3.3] implies (ii).

For a second argument, assume (i) again. Let $\text{Pow}(\{1, \dots, n\})$ denote the set of all subsets of $\{1, \dots, n\}$. It is known that L is isomorphic to an appropriate join-subsemilattice \mathfrak{F} of the powerset $(\text{Pow}(\{1, \dots, n\}); \cup)$ such that $\emptyset \in \mathfrak{F}$ and each $X \in \mathfrak{F} \setminus \{\emptyset\}$ contains an element a with the property $X \setminus \{a\} \in \mathfrak{F}$. The structure $(\{1, \dots, n\}; \mathfrak{F})$ is an *antimatroid* on the base set $\{1, \dots, n\}$ (this concept is due to R. E. Jamison-Waldner [16]), and the existence of an appropriate \mathfrak{F} follows from P. H. Edelman [12, Theorem 3.3] and D. Armstrong [4, Lemma 2.5]; see also K. Adaricheva, V. A. Gorbunov and V. I. Tumanov [3, Subsection 3.1] and G. Czédli [7, Section 7]. Now, we can assume that $L = \mathfrak{F}$. We assert that, for any $X, Y \in \mathfrak{F}$,

$$X \prec Y \quad \text{iff} \quad X \subset Y \text{ and } |Y \setminus X| = 1. \quad (4)$$

The “if” part is obvious. For the sake of contradiction, suppose $X \prec Y$ and x and y are distinct elements in $Y \setminus X$. Pick a sequence $Y = Y_0 \supset Y_1 \supset \dots \supset Y_t = \emptyset$ in \mathfrak{F} such that $|Y_{i-1} \setminus Y_i| = 1$ for $i \in \{1, \dots, t\}$. Then there is a j such that $|Y_j \cap \{x, y\}| = 1$. This gives the desired contradiction since $X \cup Y_j \in \mathfrak{F}$ but $X \subset X \cup Y_j \subset Y$.

Armed with (4), assume that $\{A = B \wedge C, B, C, D = B \cup C\}$ is a covering square in \mathfrak{F} . Note that A and $B \cap C$ can be different; however, $A \subseteq B \cap C$. By (4), there exist $u, x \in D$ such that $B = D \setminus \{u\}$ and $C = D \setminus \{x\}$. These elements are distinct since $B \neq C$. Hence $x \in B$ and, by $A \subseteq C$, $x \notin A$. Using (4) again, we obtain $A = B \setminus \{x\}$. We have seen that whenever $[A, B]$ and $[C, D]$ are consecutive prime intervals, then there is a common x such that $A = B \setminus \{x\}$ and $C = D \setminus \{x\}$. This implies that for each trajectory T of \mathfrak{F} , there exists an $x_T \in \{1, \dots, n\}$ such that $X = Y \setminus \{x_T\}$ holds for all $[X, Y] \in T$. Clearly, this implies that (iii) holds for \mathfrak{F} , and also for L .

Next, assume (iii). Since any two prime intervals of a cover-preserving diamond would belong to the same trajectory, L contains no such diamond. Again, there are two ways to conclude (i).

First, by [7, Proposition 6.1], L is isomorphic to $L_C(\vec{\pi})$ for some k and $\vec{\pi} \in S_n^{k-1}$, and we obtain from Theorem 3 that (i) holds.

Second, H. Abels [1, Theorem 3.9(a \Rightarrow b)] implies that L is a cover-preserving join-subsemilattice of a finite distributive lattice D . Thus if $x \in L \setminus \{1\}$, then the interval $[x, x^*]_L$ of L is a cover-preserving join-subsemilattice of D . Let a_1, \dots, a_t be the covers of x in L , that is, the atoms of $[x, x^*]_L$. If we had, say, $a_1 \leq a_2 \vee \dots \vee a_t$, then we would get a contradiction in D as follows: $a_1 = a_1 \wedge (a_2 \vee \dots \vee a_t) = (a_1 \wedge a_2) \vee \dots \vee (a_1 \wedge a_t) = x \wedge \dots \wedge x = x$.

Thus a_1, \dots, a_t are independent atoms in $[x, x^*]_L$. Therefore, it follows from G. Grätzer [15, Theorem 380] and the semimodularity of $[x, x^*]_L$ that the sublattice S generated by $\{a_1, \dots, a_t\}$ in L is the 2^t -element boolean lattice. In particular, $\text{length } S = t = \text{length } ([x, x^*]_L)$ since $\{x, x^*\} \subseteq S \subseteq [x, x^*]_L$. Since the embedding is cover-preserving, the length of the interval $[x, x^*]_D$ in D is also t . Hence $|\text{Ji}([x, x^*]_D)| = t$ by [15, Corollary 112], which clearly implies $|[x, x^*]_D| \leq 2^t$. Now from $[x, x^*]_L \subseteq [x, x^*]_D$ and $2^t = |S| \leq |[x, x^*]_L| \leq |[x, x^*]_D| \leq 2^t$ we conclude $[x, x^*]_L = [x, x^*]_D$. This implies that $[x, x^*]_L$ is distributive. Thus (i) holds. \square

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